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STOCHASTIC POPULATION GROWTH MODEL USING MATLAB

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ABSTRACT
The main objective of this paper is to study the stochastic population growth model, the stochastic population is Stochastic process driven by noise or Brownian motion, and we comparative between the simulation of the deterministic model and Stochastic model by MATLAB.

Keywords: *Deterministic model, Stochastic model , population growth model , Brownian.*

I. INTRODUCTION

Stochastic differential equations (SDEs) and ordinary differential equations (ODEs) now find applications in many disciplines including inter alia engineering , economics and finance , environ metrics , physics , population dynamics , biology and medicine . One particularly important application of SDEs occurs in the modeling of problems associated with water catchment and the percolation of fluid through porous / fractured structures.

Stochastic Process:
A stochastic process is a collection of random variable $\{X(t)\}_{t\in T}$ defined on a probability space $(\Omega,\mathcal{U}, P)$ and assuming values in R^n.
Note :

Fix $\omega\in\Omega$, we can consider the function $t \to X(t,\omega)$ is called the sample path (or trajectory) of X(t) .

Random variable:
Let $(\Omega,\mathcal{U}, P)$ be a probability space a mapping $X: \Omega \longrightarrow R^n$
is called an n-dimensional random variable if for each $B \in \mathcal{B}$ we have
$X^{-1}(B) \in\mathcal{U}$
we say that X is $\mathcal{U}$- measurable .

Probability space:
A triple $(\Omega,\mathcal{U}, P)$ is called a probability space , where Ω any set , $\mathcal{U}$ is a σ-algebra and P is a probability measure on $\mathcal{U}$.

Probability measure:
If Ω is a given set, $\mathcal{U}$ be a σ - algebra of sub sets of Ω , we call P: $\mu \longrightarrow [0,1]$ a probability measure provided :

1. $P(\phi)= 0, P(\Omega) = 1$
2. If $A_1, A_2, \ldots\ldots \in\mathcal{U}$, then

$$P(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} P(A_i)$$

3. If $A_1, A_2, \ldots$.. are disjoint sets in μ , then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(Ai).$$

Stochastic differential equations (SDEs) are differential equations where stochastic processes represent one or more terms and , as a consequence , the resultant solution will also be stochastic .

In 1798 the English economist Thomas Malthus proposed that a population growth at a rate proportional to its size. This leads to the same DE as in the case of radioactive decay:

$$\frac{dN(t)}{dt} = \alpha N(t) \quad t \geq o$$

Notice that the radioactive decay has the same DE as this model of population dynamics.However, in the case of the radioactive decay the solution is accurate on long time periods, while in the case of the population dynamics only on a short term, except an idealistic situation of an isolated population with unlimited resources. For a demonstration of this model see:
http://demonstrations.wolfram.com/ContinuousExponentialGrowth/

II. BROWNIAN MOTION

The standard Brownian motion (Wiener process) is stochastic process $(B(t))_{t\in\mathbb{R}+}$ such that :

- B(0) = 0 almost surely
- The sample trajectories t ⟶B(t) are continuous, with probability 1.
- For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments $B(t_1) - B(t_0)$, $B(t_2) - B(t_1)$, … , $B(t_n) - B(t_{n-1})$ are independent .
- (The Stationary normal increments) B(t) − B(s) has normal distribution with mean zero and variance t − s .

(2.1)Properties of Brownian motion :

1. B(0) = 0 w. p. 1
2. E[B(t) – B(s)] = 0
3. $E[(B(t) - B(s))^2] = t - s$
4. $E[(B(t))^2] = t$
5. $\lim_{t\to\infty} \frac{B(t)}{t} = 0$
6. B(t) – B(s) $\sim B(t-s) - B(0) = B(t-s) \sim$ N (0, t – s) .
7. For each t ≥0 , the Brownian path is almost surely not differentiable at t .

III. ITÔ INTEGRAL

For a simple process, the Itô integral $\int_0^T XdB$ is defined as a sum

$$\int_0^T X(t) \ dB(t) = \sum_{i=0}^{n-1} \xi_i \left(B(t_{i+1}) - B(t_i)\right) \text{for } 0 \leq t \leq T .$$

In general , for $0 \leq t \leq T$, if $t_k < t \leq t_{k+1}$ then

$$\int_0^t X(u)dB(u) = \sum_{j=0}^{k-1} \xi_j \left(B_{(t_{j+1})} - B_{(t_j)}\right) + \xi_i (B_{(t)} - B_{(t_k)})$$
$$= \sum_j \xi_j (B_{(t\wedge t_{j+1})} - B_{(t\wedge t_j)}) .$$

(3.1) Properties of Itô integral:

Let I(t) = $\int_0^T X(t)dB(t)$ be an Itô integral then :

1. Linearity

if I(t) $= \int_0^T X(t)dB(t)$, J(t) = $\int_0^T Y(t)d\,B(t)$ are a simple process and $\propto$ and β are constant then

$$\int_0^T \big(\alpha X(t) \pm \beta Y(t)\big)dB(t) = \propto \int_0^T X(t)dB(t) \pm \beta \int_0^T Y(t)d\,B(t) = \alpha \mathrm{I}(t) \pm \beta \mathrm{J}(t)\,.$$

2. Zero mean property

$$E\left[\int_0^T X_t\,dB(t)\right] = 0\,.$$

3. For all [a , b] $\subset [0\,,T]$

$$\int_0^T I_{[a,b]}\,(t)dB(t) = B(b) - B(a)\ \ .$$

4. Isometry property

$$E\left[\left(\int_0^T X(t)dB(t)\right)^2\right] = \int_0^T EX^2(t)dt\,.$$

5. $\int_S^T X(t)dB(t) = \int_U^T X(t)dB(t) + \int_S^U X(t)dB(t) \qquad \forall\, 0 \le S \le U \le T\,.$

6. $\int_S^T X(t)dB(t)$ is $\mathcal{U}_T -$ measurable $\forall\, 0 \le s \le T\,.$

7. Martingale property , I$(t)$is a continuous martingale .
8. Continuity , I(t) is a continuous function of the upper limit of integration .
9. Adaptedness , for each t , I(t) is $\mathcal{U}(t)$-measurable .
10. The quadratic variation accumulated up to time t by the It$\hat{o}$ integral is

$$[I,I](t) = \int_0^t X^2(u)du\,.$$

(3.2) Itô formula:

Suppose f is a function and B(t) is a standard Brownian motion . Then for every t ,

$$f\big(B(t)\big) = f\big(B(0)\big) + \int_0^t f'\big(B(s)\big)dB(s) + \frac{1}{2}\int_0^t f''\big(B(s)\big)\,ds$$

the theorem is often written in the differential form :

$$df\big(B(t)\big) = f'\big(B(t)\big)dB(t) + \frac{1}{2}f''\big(B(t)\big)dt$$

note that $f'\big(B(t)\big)$is a continuous adapted process so that stochastic integral is well defined .

IV. GENERAL STOCHASTIC AND DETERMINISTIC DIFFERENTIAL EQUATIONS

A ordinary differential equation (ODE)

$$\frac{dx(t)}{dt} = f(t,x)\ \ ,dx(t) = f(t,x)dt\,, \qquad (4.1)$$

with initial conditions $x(t_0) = x_0$ can be written in integral form

$$x(t) = x_0 + \int_0^t f(s, x(s))ds\,, \tag{4.2}$$

where $x(t) = x(t, x_0, t_0)$ is the solution with initial conditions $x(t_0)=x_0$. An example is give as

$$\frac{dx(t)}{dt} = a(t)x(t)\,, x(t_0) = x_0\ . \tag{4.3}$$

When we take the ODE (4.3) and assume that $a(t)$ is not a deterministic parameter but rather a stochastic parameter, we get a stochastic differential equation (SDE) . The stochastic parameter $a(t)$is given as

$$a(t) = f(t) + h(t)W(t)\,, \tag{4.4}$$

where$W(t)$denotes a white noise process .
Thus , we obtain

$$\frac{dx(t)}{dt} = f(t)x(t) + h(t)x(t)W(t)\,. \tag{4.5}$$

When we write (4.5) in the differential form and used$dB(t) = W(t)dt$
where$dW(t)$denotes differential form of the Brownian motion ,
we obtain :

$$dx(t) = f(t)X(t)dt + h(t)X(t)dB(t) \tag{4.6}$$

In general an SDE is given as :

$$dX(t,\omega) = f(t, X(t,\omega))dt + g(t, X(t,\omega))dB(t,w)\,, \tag{4.7}$$

where ω denotes that $X = X(t,\omega)$ is a random variable and possesses the initialcondition $X(0,\omega) = X_o$with probability one . As an example we have already encountered

$$dY(t,\omega) = \mu(t)dt + \sigma(t)dB(t)\,.$$

Furthermore ,$f(t, X(t,\omega)) \in R$, $g(t, X(t,\omega)) \in R$, and $W(t,\omega) \in R$. Similar as in (4.2) we may write (4.7) as integral equation

$$X(t,\omega) = X_0 + \int_0^t f(s, X(s,\omega))ds + \int_0^t g(s, X(s,\omega))dB(s,\omega).$$

(4.1) Solution of (SDE):
Let $x_0 \in \mathbb{R}$, $D \subseteq \mathbb{R}$ be an open set , and σ , a: $[0,\infty) \times D \rightarrow \mathbb{R}$ be continuous . A continuous and adapted stochastic process $X = (X(t))_{t\geq 0}$ is a solution of the stochastic differential equation (SDE)

$$dX(t) = \mu\,(X(t), t)dt + \sigma(X(t), t)dB(t) \text{with } X(t_0) = x_0$$

provided that the following conditions are satisfied :
1. $X(t,\omega) \in D$for all $t \geq 0$ and $\omega \in \Omega$
2. $X_0 \equiv x_0$
3. $X(t) = X(0) + \int_0^t \mu(X(s), s)ds + \int_0^t \sigma(X(s), s)dB(s) \quad for\ all\ t \geq 0$

(4.2)Existence and uniqueness:

Theorem:
For the (SDE)

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dB(t)$$

If the following conditions are satisfied :

1. Coefficients are locally Lipschitz in X uniformly in t , such that is for every T and N , there is a constant k depending only on T and N such that for all |X| , |Y| $\leq N$ and all $0 \leq t \leq T$

$$\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq k|x-y| \ ,$$

2. Coefficients satisfy the linear growth condition
$|\mu(x,t)| + |\,\sigma(x,t)| \leq k(1-|x|)$,
3. $X(0)$is independent of $(B(t), 0 \leq t \leq T)$, and $EX^2(0) < \infty$,

Then their exist a unique solution$X(t)$of the (SDE) . $X(t)$has continuous path , moreover

$$E[\sup_{0 \leq t \leq T} X^2(t)] \leq C\big(1 + EX^2(0)\big) ,$$

where constant C depend only on k and T .

If the Coefficients depend on *x* only , the conditions can be weakened .

V. POPULATION GROWTH MODEL

5.1 Deterministic population growth model:

The deterministic population growth model

$$\frac{dN(t)}{dt} = \alpha N(t)$$
$$\frac{dN(t)}{N(t)} = \alpha dt$$

The above equation is ordinary differential equation from first order ,where $\alpha$is a constant and $N(t)$is the size of the population at time *t*. we solving this equation by separation and integration :

$$\int_0^t \frac{dN(t)}{N(t)} = \int_0^t \alpha dt$$
$$LnN(t) = \alpha t$$
$$N(t) = N(0)e^{\alpha t}$$

The deterministic or ordinary model result in either exponential growth or exponential decay of the population , we can sider the linear growth model we each individual in the population at time t .

5.1.1 Example

If the number of population in first year 20 , calculate the number of populations after 10 years , (let $\alpha = 0.25$) .
Solution :

$$\frac{dN(t)}{dt} = \alpha N(t)$$
$$N(t) = N(0)e^{\alpha t}$$
$$N(t) = 20e^{0.25t}$$

In $t = 10 \;\; N(t) = N(10) = 20e^{0.25(10)}$

We will use this code in MATLAB to run simulations of the solution from above equation

```
r=input ('Enter the period');
n=input ('Enter the starting Year');
a=input ('Enter the constant');
t=0:r;
format long
p=n*exp((a)*t);
plot(t,p,'k')
```

```
xlabel('TIME')
ylabel('POPULATION')
```

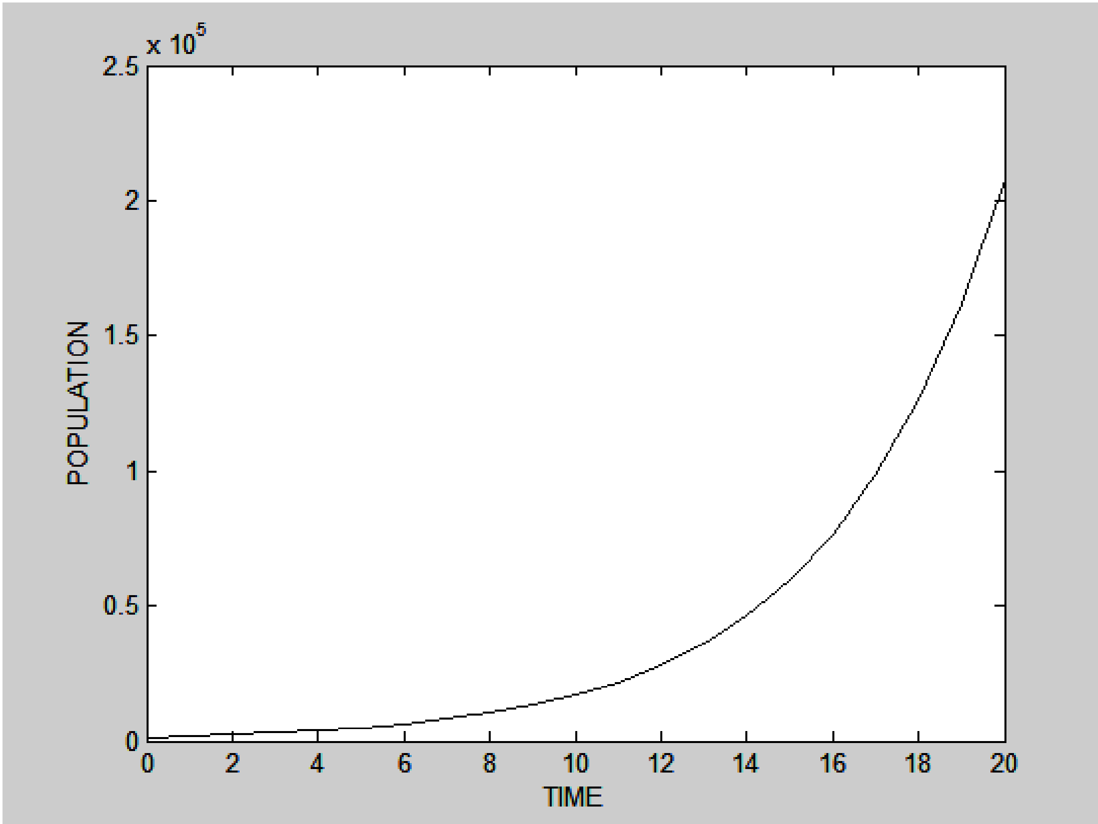

Figure (1) : Simulation asample path of deterministic model

5.2 Stochastic Population Growth Model:

Stochastic differential equations are often used in the modelling of population dynamics. For example, the *Malthusian* model of population growth (unrestricted resources) is

$$\frac{dN}{dt} = a(t)N(t) \quad , \qquad N(0) = N_0(\text{constant})$$

Where $a(t)$ is the relative rate of growth at time t .The effect of changing environmental conditions is achieved by replacing adt by a Gaussian random variable with non-zeromean adt and variance $\alpha^2 dt$to get the stochastic differential equation

$$dN(t) = aN(t)dt + \alpha N(t)dB(t) \ , N(0) = N_0$$

or

$$\frac{dN(t)}{N(t)} = adt + \alpha dB(t)$$

in which a and α(conventionally positive) are constants.

hence

$$\int_0^t \frac{dN(s)}{N(s)} = at + \alpha B(t)$$

To evaluate the integral on the left hand side we use the It$\widehat{o}$ formula for the function

$$d\big(LnN(t)\big) = \frac{1}{N(t)}dN(t) + \frac{1}{2}\left(\frac{1}{\big(N(t)\big)^2}\right)(dN_t)^2$$
$$= \frac{dN(t)}{N(t)} - \frac{1}{\big(N(t)\big)^2}\cdot \alpha^2\big(N(t)\big)^2 dt$$
$$= \frac{dN(t)}{N(t)} - \frac{1}{2}\alpha^2 dt$$

hence

$$\frac{dN(t)}{N(t)} = d\big(\mathrm{Ln}N(t)\big) + \frac{1}{2}\alpha^2 dt$$
$$\mathrm{Ln}\frac{N(t)}{N_0} = \left(a - \frac{1}{2}\alpha^2\right)t + \alpha B(t)$$

or

$$N(t) = N_0 \exp\left(\left(a - \frac{1}{2}\alpha^2\right)t + \alpha B(t)\right).$$

(5.2.1) Example:

$$dN(t) = 0.2N(t)dt + 0.25N(t)dB(t)$$

Sol :

$$N(t) = N(0)exp\left(\left[0.2 - \frac{(0.25)^2}{2}\right]t + 0.25B(t)\right)$$

We will use this code in MATLAB to run simulations of the exact solution from above equation

C:\MATLAB6p1\work\saraa.m

```
clc
clear all
nt0=20;t0=0;tf=10;dt=0.01;r=0.2;alfa=0.25;
t=t0:dt:tf;
n=max(size(t));
sqdt=sqrt(dt);
bt=zeros(n,1);
nt=zeros(n,1);
nt(1)=nt0;
for i=1:n-1
    bt(i+1)=bt(i)+sqdt*randn;
    nt(i+1)=nt(1)*exp((r-(1/2*(alfa)^2))*t(i)+alfa*bt(i+1));
end
plot(t,nt,'y')
title('Exact')
xlabel('Time')
ylabel('Population')
legend('run1','run2','run3','run4','run5')
hold on
```

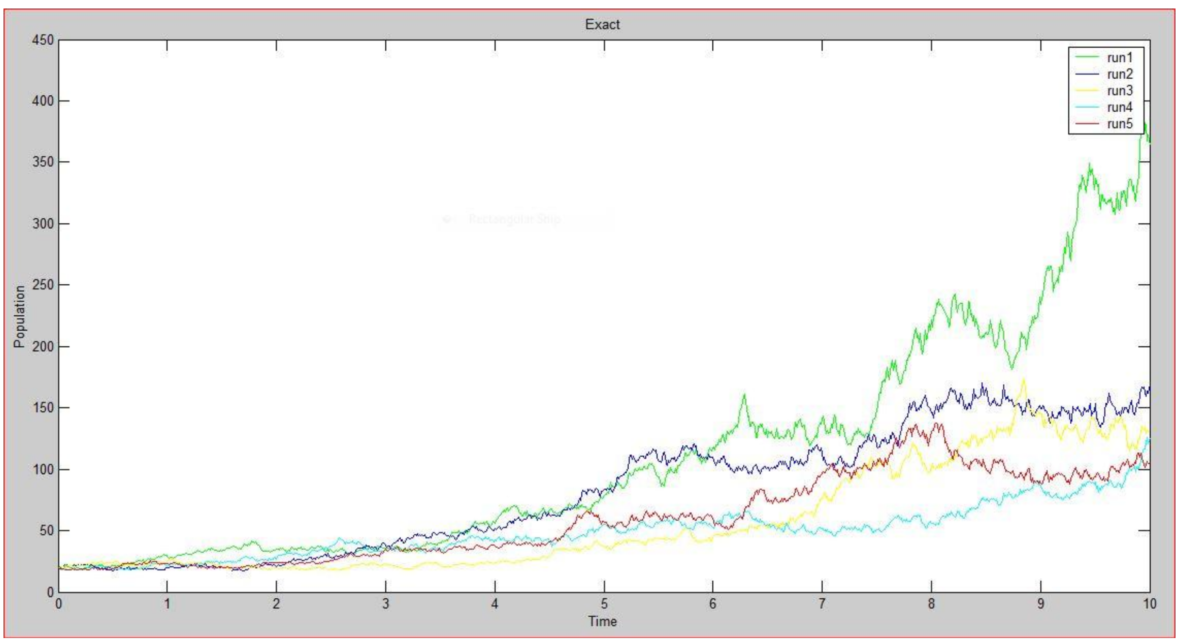

Figure (2) : simulation a sample path of stochastic model

This five independent runs of the stochastic model for population growth over the time interval [0 , 10] with $dt = 0.01$ and $N(0) = 20$ The parameter values , $r = 0.2$ and $\alpha = 0.25$

VI. CONCLUSION

In order to build the Stochastic Differential Equations models , one can add a random perturbation in the classic ordinary differential equations model assuming that the dynamics are partly driven by noise ,we note that in figure (1) which simulation equation of deterministic model the trajectory is smooth , but in figure (2) which simulation equation of stochastic population model the trajectory is not smooth.

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